Electric-magnetic duality and supersymmetry in stringy black holes

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We present a generalization of the $U(1)^2$ charged dilaton black hole family whose main feature is that both U(1) fields have electric and magnetic charges, the axion field still being trivial. We show the supersymmetry of these solutions in the extreme case in which the corresponding generalization of the Bogomolnyi bound is saturated and a naked singularity is on the verge of being visible to external observers. Then we study the action of a subset of the SL(2,R) group of electric-magnetic duality rotations that generates a nontrivial axion field on those solutions. This group of transformations is an exact symmetry of the $N=4$, $d=4$ ungauged supergravity equations of motion. It has been argued recently that it could be an exact symmetry of the full effective string theory. The generalization of the Bogomolnyi bound is invariant under the full SL(2,R) and the solutions explicitly rotated are shown to be supersymmetric if the originals are. We conjecture that any SL(2,R) transformation will preserve supersymmetry.

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I. INTRODUCTION

Nowadays, much effort is being devoted to the study of black holes in string theory from different perspectives as an attempt to elucidate the properties of the quantum gravity theory which is embedded in it. Many of the unusual features of this theory have their origin in the atypical coupling between the graviton and the dilaton fields which has become the particular signature of string theory in toy and low-energy models.

Classical solutions of the low-energy effective actions of string theory in four dimensions with a nontrivial dilaton field are very interesting in this framework [1,2]. They exhibit the classical properties of the dilaton interactions and provide the starting point for the study of semiclassical (quantum) behavior (Hawking radiation, etc.). This and the fact that they may inherit properties of the full theory (some unbroken supersymmetries [3], some properties of the spectrum [4], etc.) justify their study.

Another stimulating aspect of these solutions is that they can always be seen as solutions of the general relativity theory in the presence of exotic matter. This naturally suggests the embedding in a locally supersymmetric theory also hinted by string theory. Local supersymmetry techniques might be major tools in the study of general relativity. As an example let us recall Witten's proof of the positive energy theorem in general relativity [5] using techniques borrowed from supergravity (see [6] and references therein). After Witten's work, spinor techniques related to supergravity were used to prove other results of the same kind, inequalities relating the (asymptotically defined) charges of spacetimes for which some positivity condition of the energy-momentum tensor holds (see, for example, [7] and the review paper [8]). These inequalities are also known in the context of supersymmetry as Bogomolnyi bounds [9,10] and solutions that saturate them have special supersymmetric and geometrical properties. It has been noticed many times that "extreme" black holes (i.e., on the verge of showing a naked singularity) admit unbroken supersymmetries and saturate a Bogomolnyi-type bound (see [3] and references therein). Therefore, supersymmetry seems to act as a cosmic censor. Some exceptions are known in nonasymptotically flat backgrounds [11], but we still believe that some connection may exist between cosmic censorship and supersymmetry in some restricted subset of asymptotically flat spacetimes. This idea is one of the motivations for looking at the supersymmetry properties of these solutions.

Finally, methods for generating new solutions from those already known have recently been described in [4,12,13]. They take advantage of the symmetries of the equations of motion. From the viewpoint of string theory the noninvariance of the effective action is not an issue because the only virtue of the action is that its extrema verify those equations of motion. We will be interested in the latter of these methods, which consists in performing a combination of "dual rotations" and constant shifts of the axion field of a given solution. This operation rotates electric and magnetic charges and, as we will see, dilaton and axion charges into each other so we can find new solutions with nontrivial axion fields starting from solutions with a nontrivial dilaton. These transformations were known to generate the SL(2,R) group of exact symmetries of the equations of motion of $N=4$, $d=4$ ungauged supergravity (the SU(1,1) group of Ref. [16]) whose action coincides with part of the effective action of the dimensionally reduced action of the heterotic string. In Ref. [4], Sen has shown that this is still a symmetry of the equations of motion when one includes scalar and

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vector terms coming from the compactification process and has conjectured that SL(2,Z) might be a symmetry of the full effective string theory after the axion shift symmetry is broken by instantons. The preservation of unbroken supersymmetries of a quite general family of extreme spherical charged dilaton black holes by a subgroup of these transformations is one of the main results of this paper.

The family of charged dilaton black holes that we are going to study here generalizes the one recently discussed in [3] (originally discovered by Gibbons [14] and discussed by Gibbons and Maeda [15]). It will be introduced in Sec. II. The main feature of these solutions is the presence of magnetic and electric charges corresponding to each of the U(1) fields, while the axion remains trivial. In Sec. III we will focus on the supersymmetry properties of the extreme solutions and will show the existence of supercovariantly constant spinors in those backgrounds and the saturation of a Bogomolnyi-type bound. In Sec. IV we will study the effect of the subgroup of SL(2,R) used in Ref. [13] on the charges and will see that the corresponding Bogomolnyi-type bound is left invariant. This will be shown in Sec. V to mean that backgrounds obtained by rotating supersymmetric ones, are also supersymmetric. Since the full SL(2,R) preserves the Bogomolnyi bound, we conjecture that the same will happen with any of those transformations. A discussion and further comments are the content of the last section and some conventions are specified in the Appendix.

II. DOUBLY CHARGED SOLUTIONS

The truncation of the action of the SU(4) version of $N=4$, $d=4$ ungauged supergravity we are going to work with is (our conventions are the same as in [3])

$$S_{\mathrm{SU}(4)}=\int dx^4\sqrt{-g}\,[-R+2(\partial\phi)^2+\tfrac{1}{2}e^{4\phi}(\partial a)^2 -e^{-2\phi}(F^2+G^2) +ia(F*F+G*G)]\ . \tag{1}$$

The equations of motion read

$$\begin{aligned}
&\nabla_\mu(e^{-2\phi}F^{\mu\nu}-ia*F^{\mu\nu})=0\ ,\\
&\nabla_\mu *F^{\mu\nu}=0\ ,\\
&\nabla_\mu(e^{-2\phi}G^{\mu\nu}-ia*G^{\mu\nu})=0\ ,\\
&\nabla_\mu *G^{\mu\nu}=0\ ,\\
&\nabla^2\phi-\tfrac{1}{2}e^{4\phi}(\partial a)^2-\tfrac{1}{2}e^{-2\phi}(F^2+G^2)=0\ ,\\
&\nabla^2 a+4\partial_\mu\phi\partial^\mu a-ie^{-4\phi}(F*F+G*G)=0\ ,\\
&R_{\mu\nu}+2\partial_\mu\phi\partial_\nu\phi+\tfrac{1}{2}e^{4\phi}\partial_\mu a\partial_\nu a\\
&\quad -2e^{-2\phi}[(F_{\mu\rho}F^{\rho}_{\nu}-\tfrac{1}{4}g_{\mu\nu}F^2+G_{\mu\rho}G^{\rho}_{\nu}-\tfrac{1}{4}g_{\mu\nu}G^2)]=0\ .
\end{aligned}\tag{2}$$

F and G are two U(1) fields and ϕ and a are the dilaton and the axion, respectively. The latter is a pseudoscalar. These can be combined in a single complex scalar $z=e^{-2\phi}-ia$ (z is $-i$ times λ in Ref. [13]) using the self- and anti-self-dual parts of F and G, which makes the duality rotations easier to describe. In terms of z the action and equations of motion are

$$S_{\mathrm{SU}(4)}=\int dx^4\sqrt{-g}\left\{-R+\frac{2\partial_\mu z\partial^\mu\bar{z}}{(z+\bar{z})^2} -\{z[(F^+)^2+(G^+)^2]+\text{c.c.}\}\right\}\ , \tag{3}$$

$$\begin{aligned}
&\nabla_\mu(zF^{+\mu\nu}+\text{c.c.})=0\ ,\\
&\nabla_\mu(F^{+\mu\nu}-\text{c.c.})=0\ ,\\
&\nabla_\mu(zG^{+\mu\nu}+\text{c.c.})=0\ ,\\
&\nabla_\mu(G^{+\mu\nu}-\text{c.c.})=0\ ,\\
&\nabla^2 z-2\frac{(\partial z)^2}{z+\bar{z}}+\frac{(z+\bar{z})^2}{2}[(F^-)^2+(G^-)^2]=0\ ,\\
&\nabla^2\bar{z}-2\frac{(\partial\bar{z})^2}{z+\bar{z}}+\frac{(z+\bar{z})^2}{2}[(F^+)^2+(G^+)^2]=0\ ,\\
&R_{\mu\nu}+\frac{2\partial_{(\mu}z\partial_{\nu)}\bar{z}}{(z+\bar{z})^2}-(z+\bar{z})(F_{\mu\rho}F^{\rho}_{\nu}-\tfrac{1}{4}g_{\mu\nu}F^2\\
&\quad +G_{\mu\rho}G^{\rho}_{\nu}-\tfrac{1}{4}g_{\mu\nu}G^2)=0\ .
\end{aligned}\tag{4}$$

The SO(4) version is obtained by first substituting G by $\tilde{G}$ defined by

$$G^{+}_{\mu\nu}=-iz^{-1}\tilde{G}^{+}_{\mu\nu} \tag{5}$$

in the equations of motion. Then it is still necessary to reverse the sign of the $\tilde{G}$ terms in the action.

It is just a matter of calculation to see that a solution is provided by

$$\begin{aligned}
&ds^2=e^{2U}dt^2-e^{-2U}dr^2-R^2d\Omega^2\ ,\\
&e^{2\phi}=e^{2\phi_0}\frac{r+\Sigma}{r-\Sigma}\ ,\\
&a=a_0\ ,\\
&F=Q_F\frac{e^{2(\phi-\phi_0)}}{R^2}dt\wedge dr-P_F\sin\theta\,d\theta\wedge d\phi\ ,\\
&G=Q_G\frac{e^{2(\phi-\phi_0)}}{R^2}dt\wedge dr-P_G\sin\theta\,d\theta\wedge d\phi\ ,
\end{aligned}\tag{6}$$

where the different functions and constant are

$$\begin{aligned}
&e^{2U}=\frac{(r-r_+)(r-r_-)}{R^2}\ ,\\
&R^2=r^2-\Sigma^2\ ,\\
&\Sigma=e^{-2\phi_0}\frac{(P_F^2+P_G^2)-(Q_F^2+Q_G^2)}{2M}=\Sigma_F+\Sigma_G\ ,\\
&r_\pm=M\pm r_0\ ,\\
&r_0^2=M^2+\Sigma^2-e^{-2\phi_0}(Q_F^2+P_F^2+Q_G^2+P_G^2)\ .
\end{aligned}\tag{7}$$

$Q_{F(G)}$ and $P_{F(G)}$ are the $F(G)$ field electric and magnetic charges, respectively. $\Sigma_{F(G)}$ is the $F(G)$ contribution to the dilaton charge Σ. For this family of solutions they are not independent quantities. ϕ_0 and a_0 are the asymptotic (constant) value of the dilaton and axion. Only if the electric and magnetic charges satisfy

$$Q_F P_F + Q_G P_G = 0 \tag{8}$$

does the axion equation hold and we have a solution of the equations of motion.

Motivated by what follows, we define the axion charge Δ as a quantity depending on the electric and magnetic charges as well on the asymptotic value of the dilaton in this way:

$$\Delta = -e^{-2\phi_0} \frac{Q_F P_F + Q_G P_G}{M} = \Delta_F + \Delta_G \ , \tag{9}$$

so we can interpret (8) as the condition of null axion charge. The charge of the complex scalar is then

$$\Upsilon = \Sigma - i\Delta = \frac{(P_F + iQ_F)^2}{2M} + \frac{(P_G + iQ_G)^2}{2M} = \Upsilon_F + \Upsilon_G \ , \tag{10}$$

and can correspondingly be expressed in terms of the (complex) charge of the self-dual parts of F and G:

$$\Gamma_{F(G)} = \tfrac{1}{2}(Q_{F(G)} + iP_{F(G)}) \ , \tag{11}$$

$$\Upsilon_{F(G)} = -2\frac{(\overline{\Gamma_{F(G)}})^2}{M} \ . \tag{12}$$

All these relations are consistent with the definitions of the charges in terms of the asymptotic behavior ($r \to \infty$) of the different fields:

$$\begin{aligned}
g_{tt} &\sim 1 - \frac{2M}{r} \ , \\
\phi &\sim \phi_0 + \frac{\Sigma}{r} \ , \quad \left[e^{-2\phi} \sim e^{-2\phi_0}\left[1 - \frac{2\Sigma}{r}\right] \right] \ , \\
a &\sim a_0 - e^{-2\phi_0}\frac{2\Delta}{r} \ , \\
z &\sim z_0 - e^{-2\phi_0}\frac{2\Upsilon}{r} \ , \\
F_{\mathrm{tr}} &\sim \frac{Q_F}{r} \ , \\
{*F}_{\mathrm{tr}} &\sim i\frac{P_F}{r} \ , \\
F^{+}_{\mathrm{tr}} &\sim \frac{\Gamma_F}{r} \ .
\end{aligned} \tag{13}$$

Now let us briefly describe the main properties of this family of solutions. Setting $P_F = Q_G = 0$ we get that of Ref. [3] which essentially is one of the solutions in [14,15] generalized to $\phi_0 \neq 0$. Reissner-Nordström black holes and the charged dilaton black holes described in [1] were already included in [14,15]. The axion charge was zero in the case considered there because the contributions of the fields F and G to it were identically zero: $\Delta_F = \Delta_G = 0$. Now, in the more general case considered here both contributions cancel each other: $\Delta_F = -\Delta_G$.

We remark that our solutions cannot be obtained from the solutions considered in Refs. [3,14,15] by means of an SL(2,R) transformation of the type we are going to consider in Sec. III.

Notice that now the metric, dilaton, and axion fields are essentially the same as in those references, the only difference being that we have to replace every single electric or magnetic charge in their expressions by the orthogonal sum of our pair of charges. All the properties that depend on the metric (Hawking temperature, etc.) can be found in this way from those of the case $P_F = Q_G = 0$. This means, in particular, that we again have spherical black holes with two horizons at $r = r_\pm$ which coincide when $r_0 = 0$.

Black holes with $r_0 = 0$ are called extreme because they are on the verge of having a naked singularity. No solution with a smaller mass and the same charges is regular outside a horizon. Since there are no U(1) charged particles in our theory, this means that the evaporation of any regular black hole ($r_0 \geq 0$) should stop when they become extreme. It was proven in [3] that the extreme black holes with $P_F = Q_G = 0$ have unbroken supersymmetries and saturate the Bogomolnyi-type bound:

$$M^2 + \Sigma^2 \geq e^{-2\phi_0}(Q_F^2 + P_G^2) \ . \tag{14}$$

In our case, the condition $r_0^2 \geq 0$ gives

$$M^2 + \Sigma^2 + \Delta^2 \geq e^{-2\phi_0}(Q_F^2 + Q_G^2 + P_F^2 + P_G^2) \ , \tag{15}$$

which is more easily written by using the complex charges defined above:

$$M^2 + |\Upsilon|^2 \geq 4e^{-2\phi_0}(|\Gamma_F|^2 + |\Gamma_G|^2) \ . \tag{16}$$

We have included the (vanishing) axion charge with the sign with which it will appear later. This bound is also saturated for extreme black holes, and so we expect them to have unbroken supersymmetries. Our next task will be to prove that this property actually holds for the more general class of extreme black holes we are dealing with. We will not derive the Bogomolnyi-type bound (16) from the supersymmetry algebra here, though.

III. SUPERSYMMETRY OF THE DOUBLY CHARGED SOLUTIONS

Now we want to consider our solution as the bosonic part of a solution of the full supersymmetric theory in which all the fermionic fields are zero. A natural question to ask is whether this solution is invariant under some local supersymmetry transformations. The variations of the bosonic fields are proportional to the fermionic fields and so they are obviously zero. The variations of the fermionic fields (gravitinos and dilatinos) are proportional to the bosonic fields and so, only for very special backgrounds there will be a finite number of transformations leaving them partially invariant, i.e., obeying $\delta_\epsilon \psi_{\mu I} = 0$ and $\delta_\epsilon \Lambda_I = 0$ for some of the SO(4) indices I. Finding these transformations is equivalent to finding the spinors ϵ_I (ϵ^I) that satisfy the differential equations

$$\nabla_\mu\epsilon_I-\frac{i}{4}e^{2\phi}\partial_\mu a\,\epsilon_I -\frac{e^{-\phi}}{2\sqrt{2}}\sigma^{\rho\sigma}(F_{\rho\sigma}\alpha_{IJ}+z^{-1}\tilde{G}_{\rho\sigma}\beta_{IJ})\gamma_\mu\epsilon^J=0\ , \tag{17}$$

$$-\gamma^\mu\left[\partial_\mu\phi+\frac{i}{2}e^{2\phi}\partial_\mu a\right]\epsilon_I +\frac{e^{-\phi}}{\sqrt{2}}\sigma^{\rho\sigma}(F_{\rho\sigma}\alpha_{IJ}-\bar{z}^{-1}\tilde{G}_{\rho\sigma}\beta_{IJ})\epsilon^J=0\ , \tag{18}$$

respectively, in the SO(4) formulation that we use here for convenience.

On the other hand, if some ϵ_I's exist that are asymptotically constant in the limit $r\to\infty$, we can speak about asymptotic, rigid (i.e., nonlocal) supersymmetry and establish Bogomolnyi-type bounds concerning the (asymptotically defined) charges of the background. Our purpose here is to find these kinds of solutions among the family described in Sec. I. The construction of the (asymptotic) supersymmetry subalgebra that leaves invariant the states representing the backgrounds and which is necessary to derive the Bogomolnyi bound will not be explicitly made here for this would repeat the work done in [3]. Only the new basis of supersymmetry will be shown. Accordingly we will impose the condition of time independence on the solutions for which we are looking:

$$\partial_t\epsilon_I=0\ . \tag{19}$$

We take $a_0=0$ for simplicity.[1] After some algebra, adding and subtracting the equations $\delta_\epsilon\psi_{tI}=0$ and $\delta_\epsilon\Lambda_I$ we arrive at the equations

$$(\partial_r e^{U+\phi})\epsilon_I=-\sqrt{2}e^{-2\phi}\frac{e^{2\phi}}{R^2}(Q_F\alpha_{IJ}+iQ_G\beta_{IJ})\gamma^0\epsilon^J\ , \tag{20}$$

$$(\partial_r e^{U-\phi})\epsilon_I=-i\sqrt{2}\frac{e^{-2\phi}}{R^2}(P_F\alpha_{IJ}+iP_G\beta_{IJ})\gamma^0\epsilon^J\ . \tag{21}$$

Using them in $\delta_\epsilon\psi_{rI}=0$, $\delta_\epsilon\psi_{\theta I}=0$, and $\delta_\epsilon\psi_{\varphi I}=0$ we obtain

$$\partial_r(e^{-(1/2)U}\epsilon_I)=0\ , \tag{22}$$

$$\partial_\theta\epsilon_I-\frac{i}{2}\partial_r(\mathrm{Re}^U)\gamma^3\gamma^0\epsilon_I=0\ , \tag{23}$$

$$\partial_\varphi\epsilon_I-\frac{i}{2}\sin\theta\,\partial_r(\mathrm{Re}^U)\gamma^2\gamma^0\epsilon_I+\cos\theta\,\gamma^1\gamma^0\epsilon_I=0\ . \tag{24}$$

The solution to the first of these equations is

$$\epsilon_I=e^{(1/2)U}\hat{\epsilon}_I\ , \tag{25}$$

where $\hat{\epsilon}_I$ is a function of only θ and φ. The other two equations have the same form in terms of $\hat{\epsilon}_I$. Now, if we apply ∂_r to any of them we get the integrability condition

$$\partial_r^2(\mathrm{Re}^U)=0\Longrightarrow \mathrm{Re}^U=br+d\ , \tag{26}$$

which is tantamount to saying that is has to be possible to write the metric in isotropic form:

$$ds^2=e^{2U}dt^2-e^{-2U}(d\rho^2+\rho^2d\Omega^2)\ . \tag{27}$$

This happens only when $r_0=0$, the extreme case. From now on we will write

$$\partial_r(\mathrm{Re}^U)=1\ . \tag{28}$$

Now it is easy to check that the two last angular equations are solved by

$$\hat{\epsilon}_I=e^{(i/2)\gamma^3\gamma^0\theta}e^{(i/2)\gamma^1\gamma^0\varphi}\epsilon_{I_0}\ , \tag{29}$$

where ϵ_{I_0} is a constant spinor. The exponentials can be calculated using the explicit expressions for γ matrices written in the Appendix. The result is

$$e^{(i/2)}\gamma^i\gamma^0x=\cos(x/2)+i\gamma^i\gamma^0\sin(x/2)\ . \tag{30}$$

Then, if a solution exists, it must have the form

$$\epsilon_I(r,\theta,\varphi)=e^{(1/2)U}e^{(i/2)\gamma^3\gamma^0\theta}e^{(i/2)\gamma^1\gamma^0\varphi}\epsilon_{I_0}\ . \tag{31}$$

Observe that, as we wanted, these spinors are asymptotically constant.[2]

So far, we have solved the equations involving only the negative chirality spinors. Now we have to solve the equations relating positive and negative chirality spinors, namely, (20) and (21) and also the Majorana condition which, with our conventions reads

$$\chi_I=\gamma^2(\chi^I)^*\ . \tag{32}$$

Making the same choice of α_{IJ} and β_{IJ} as in Ref. [3], consistency between Eqs. (20), (21), and (32) is achieved if

$$\left|\sqrt{2}\frac{e^{2(\phi-\phi_0)}}{R^2}(Q_F+iQ_G)\right|=\partial_r e^{U+\phi}\ , \tag{33}$$

$$\left|\sqrt{2}\frac{e^{-2\phi}}{R^2}(P_F+iP_G)\right|=\partial_r e^{U-\phi}\ , \tag{34}$$

which is true only in the extreme case. Within this family of solutions, only extreme black holes can have unbroken supersymmetries. Now let us define for ($U\neq\pm\phi$) the two complex phases

$$\xi=\frac{Q_F+iQ_G}{(Q_F^2+Q_G^2)^{1/2}}\ , \tag{35}$$

[1]Actually, a_0 does not enter in the SU(4) supersymmetry rules that we could have used.

[2]The angular dependence can be eliminated by a change of coordinates. Compare, for instance, our solution with Eq. (105) of Ref. [3].

$$\eta = \frac{P_F + iP_G}{(P_F^2 + P_G^2)^{1/2}} , \tag{36}$$

where we choose the positive branch of the square root. These complex phases were just plus or minus signs in the case covered in Ref. [3]. In the $I,J=1,2$ sector we have

$$\begin{aligned} &\partial_r e^{U+\phi}(\epsilon_1 + \xi^* \gamma^0 \epsilon^2) = 0 , \\ &\partial_r e^{U-\phi}(\epsilon_1 + i\eta^* \gamma^0 \epsilon^2) = 0 , \end{aligned} \tag{37}$$

and in the $I,J=3,4$ sector we have

$$\begin{aligned} &\partial_r e^{U+\phi}(\epsilon_3 + \xi \gamma^0 \epsilon^4) = 0 , \\ &\partial_r e^{U-\phi}(\epsilon_3 + i\eta \gamma^0 \epsilon^4) = 0 . \end{aligned} \tag{38}$$

Notice that these equations have solutions only for the *pairs* of spinors ϵ_1, ϵ_2 and/or ϵ_3, ϵ_4. The four spinors have to be of the form given by (31), the difference being the constant spinor we pick. The previous equations are, then, two relations between ϵ_{1_0} and ϵ_{2_0} and two relations between ϵ_{3_0} and ϵ_{4_0}. Both relations must be compatible.

Then, unbroken supersymmetry in the 1,2 sector means

$$\begin{aligned} &\epsilon_1 = -\xi^* \gamma^0 \epsilon^2 , \\ &i\eta^* = \xi^* , \end{aligned} \tag{39}$$

which implies, for some positive finite constant μ,

$$\begin{aligned} &Q_F = \mu P_G , \\ &Q_G = -\mu P_F , \end{aligned} \tag{40}$$

that is, $\Delta = 0$. Only ϵ_{1_0} or ϵ_{2_0} can be chosen arbitrarily but subject to the chirality constraint). This means two complex (four real) arbitrary constants, that is, unbroken $N=1$ supersymmetry.

In the 3,4 sector, unbroken supersymmetry means

$$\begin{aligned} &\epsilon_3 = -\xi \gamma^0 \epsilon^4 , \\ &i\eta = \xi , \end{aligned} \tag{41}$$

which implies, for some positive finite constant μ,

$$\begin{aligned} &Q_F = -\mu P_G , \\ &Q_G = \mu P_F . \end{aligned} \tag{42}$$

We obtain again the condition $\Delta = 0$ and unbroken $N=1$ supersymmetry.

Note that for $U \neq \pm\phi$ only one of the pairs of supersymmetries 1,2 or 3,4 can be unbroken at once. We can only have unbroken $N=1$ supersymmetry.

We can define a new basis for the supersymmetries:

$$\begin{aligned} &\epsilon_\xi^{ij} = \epsilon_j + \xi \gamma^0 \epsilon^i , \\ &\epsilon_{\xi ij} = \epsilon^j + \xi \gamma^0 \epsilon_i , \\ &\epsilon_{\xi^*}^{ij} = \epsilon_j + \xi^* \gamma^0 \epsilon^i , \\ &\epsilon_{\xi^* ij} = \epsilon^j + \xi^* \gamma^0 \epsilon_i , \end{aligned} \tag{43}$$

where the pair ij takes the values 12 and 34. When $\mathrm{sgn}(Q_F) = \mathrm{sgn}(P_G)$ [which implies $\mathrm{sgn}(Q_G) = -\mathrm{sgn}(P_F)$ for Δ to be zero] ϵ_ξ^{12} and $\epsilon_{\xi 12}$ are unbroken. If

$$\mathrm{sgn}(Q_F) = -\mathrm{sgn}(P_G)[\mathrm{sgn}(Q_G) = \mathrm{sgn}(P_F)]$$

then $\epsilon_{\xi^*}^{34}$ and $\epsilon_{\xi^* 12}$ are unbroken. Obviously, the basis is different for each specific case.

If $U = +\phi$, then $P_F = P_G = 0$ (purely electric case). We have only one relation between spinors in each sector which implies no extra compatibility condition on the charges. This means that we have unbroken supersymmetry both in the 1,2 and in the 3,4 sectors, and, as a consequence, unbroken $N=2$ supersymmetry. The ϵ_ξ^{ij} and $\epsilon_{\xi ij}$ $ij = 12, 34$ are the unbroken supersymmetries in this case.

Let us summarize the results obtained in this section. Doubly charged extreme solutions with $U \neq \pm\phi$ are $N=1$ supersymmetric. The cases $U = \pm\phi$ (purely electric and purely magnetic, respectively) are $N=2$ supersymmetric.

IV. SL(2,R) ROTATION OF DOUBLY CHARGED DILATON BLACK HOLES

In Ref. [13] it was found that the equations of motion of the action[3] (3) were almost invariant under an SL(2,R) group of transformations generated by performing alternatively Peccei-Quinn shifts of the axion by a constant $a \to a + c$,

$$z \to z - i\beta , \tag{44}$$

and the duality transformation which in the absence of an axion is the transformation $\phi \to -\phi$ that trades electric for magnetic solutions:

$$\begin{aligned} &z \to 1/z , \\ &F^+ \to -izF^+ , \\ &F^- \to i\bar{z}F^- . \end{aligned} \tag{45}$$

The action of a general SL(2,R) transformation on z and F is[4]

$$\begin{aligned} &z \to \frac{\alpha z - i\beta}{i\gamma z + \delta} , \quad \alpha\delta - \beta\gamma = 1 , \\ &F^+ \to -(i\gamma z + \delta) F^+ . \end{aligned} \tag{46}$$

As it is explained in Ref. [4] this is an exact symmetry of the equations of motion because the offending extra term that appears in the last equation of motion (4) is proportional to

$$F_{(\alpha|\rho|} {}^*F_{\beta)}{}^\rho - \tfrac{1}{4} g_{\alpha\beta} F {}^*F \tag{47}$$

[3]The inclusion of the second vector field G does not make any qualitative difference.

[4]For any $M \in \mathrm{SL}(2,R)$, M and $-M$ have the same action on z, but not on the vector fields that transform with different signs under M and $-M$. This sign is not important from the point of view of the invariance of the equations of motion. This is why we keep here the sign conventionally chosen in the literature in (46) even though it is not completely consistent.

(and an analogous term for G), which vanishes identically in four dimensions. (One has to calculate each component of this expression or use the Newmann-Penrose formalism to prove it.) The existence of this group of exact invariances was already known in the context of $N=4$, $d=4$ ungauged supergravity (the SU(1,1) group of Ref. [16]). In this respect, the novelty in [4] is the inclusion of more terms in the action coming from the compactification of the 6 extra dimensions of heterotic superstring theory on a torus.

In this section we are going to use the reduced subset of $SL(2,R)$ transformations that was used in Ref. [13] to generate backgrounds with nontrivial axion field. They consist of a shift of the axion by a constant $\beta=c$ followed by the duality transformation (46) and a rescaling by the normalization constant N:

$$\begin{aligned}
&z \to z' = \frac{-i(-N)^{1/2}}{iN^{-1/2}z + cN^{-1/2}} ,\\
&F^+ \to F^{+'} = -(iN^{-1/2}z + cN^{-1/2})F^+ ,\\
&F^- \to F^{-'} = -(-iN^{-1/2}bz + cN^{-1/2})F^- ,
\end{aligned} \tag{48}$$

and the same for G. Using our definitions of the charges (13) we see that the transformations described above act on the charges and asymptotic values of any background in the following way:

$$\begin{aligned}
a_0' &= \frac{-Ne^{2\phi_0}}{(a_0+c)^2 e^{2\phi_0} + e^{-2\phi_0}}(a_0+c) ,\\
e^{-2\phi_0'} &= \frac{N}{(a_0+c)^2 e^{2\phi_0} + e^{-2\phi_0}} ,\\
Q_F' &= \frac{-(a_0+c)}{N^{1/2}} Q_F + \frac{e^{-2\phi_0}}{N^{1/2}} P_F ,\\
P_F' &= -\frac{e^{-2\phi_0}}{N^{1/2}} Q_F - \frac{-(a_0+c)}{N^{1/2}} P_F ,\\
\Sigma_F' &= \frac{(a_0+c)^2 e^{2\phi_0} - e^{-2\phi_0}}{(a_0+c)^2 e^{2\phi_0} + e^{-2\phi_0}} \Sigma_F\\
&\quad - \frac{2(a_0+c)}{(a_0+c)^2 e^{2\phi_0} + e^{-2\phi_0}} \Delta_F ,\\
\Delta_F' &= \frac{2(a_0+c)}{(a_0+c)^2 e^{2\phi_0} + e^{-2\phi_0}} \Sigma_F\\
&\quad + \frac{(a_0+c)^2 e^{2\phi_0} - e^{-2\phi_0}}{(a_0+c)^2 e^{2\phi_0} + e^{-2\phi_0}} \Delta_F .
\end{aligned} \tag{49}$$

Similar expressions hold for G charges.

In [13] N was chosen to preserve $|\Gamma_F|^2$. However, what we want to preserve is actually $e^{-2\phi_0}|\Gamma_F|^2$ because that is what appears in the bound (16). For any value of N, $|\Upsilon_{F(G)}|^2$ and the combination $e^{-2\phi_0}|\Gamma_{F(G)}|^2$ are invariant under (49). In fact, these transformations can be written as rotations:

$$\begin{aligned}
&e^{-\phi_0'}\Gamma' = e^{-i\alpha}(e^{-\phi_0}\Gamma) ,\\
&\Upsilon' = e^{+2i\alpha}\Upsilon ,\\
&\sin\alpha = \frac{e^{-\phi_0}}{[(a_0+c)^2 e^{2\phi_0} + e^{-2\phi_0}]^{1/2}} ,\\
&\cos\alpha = \frac{-(a_0+c)}{[(a_0+c)^2 e^{2\phi_0} + e^{-2\phi_0}]^{1/2}} .
\end{aligned} \tag{50}$$

Therefore, the Bogomolnyi bound (16) is invariant. If we rotate a background in which it is saturated, we will obtain another one with the same property. The supersymmetry property associated with it should also be preserved.

In fact, it does not take much effort to see that the Bogomolnyi bound is invariant under the full $SL(2,R)$ group. As a consequence, it should transform supersymmetric backgrounds into supersymmetric backgrounds. Here we will focus on the subset of transformations described above and will prove the supersymmetry of the transformed solutions in the next section. The general case is not much more illuminating and should follow straightforwardly. The results will be presented elsewhere.

The Bogomolnyi bound can also be interpreted as a condition of equilibrium of forces. The electric-magnetic dual rotations preserve this equilibrium not only by permutating the charges. Notice the curious interplay between dilaton and electromagnetic forces. The scaling of the electric and magnetic charges in a $SL(2,R)$ transformation is absorbed in the scaling of e^{ϕ_0}, the string coupling constant at infinity. The existence of doubly charged multi-extreme-black-hole solutions in which the Bogomolnyi bound is clearly seen as a condition of equilibrium of forces is very likely. We will not try to study them here.

V. SUPERSYMMETRY OF THE ROTATED SOLUTIONS

Here we will present the calculations quite schematically for we followed exactly the same steps as in Sec. III to find the $N=4$ Killing spinors. The equations analogous to (20), (21), (22), (23), and (24) are, respectively,

$$(\partial_r e^{U+\phi})\epsilon_I = \sqrt{2} e^{i\alpha} \frac{e^{2(\phi-\phi_0)}}{R^2}(Q_F\alpha_{IJ} + iQ_G\beta_{IJ})\gamma^0\epsilon^J , \tag{51}$$

$$(\partial_r e^{U-\phi})\epsilon_I = i\sqrt{2} e^{i\alpha} \frac{e^{-2\phi}}{R^2}(P_F\alpha_{IJ} + iP_G\beta_{IJ})\gamma^0\epsilon^J , \tag{52}$$

$$\partial_r(e^{-(1/2)(U+i\alpha)}\epsilon_I) = 0 , \tag{53}$$

$$\partial_\theta\epsilon_I - \frac{i}{2}\partial_r(\mathrm{Re}^U)\gamma^3\gamma^0\epsilon_I = 0 , \tag{54}$$

$$\partial_\varphi\epsilon_I - \frac{i}{2}\sin\theta\,\partial_r(\mathrm{Re}^U)\gamma^2\gamma^0\epsilon_I + \cos\theta\,\gamma^1\gamma^0\epsilon_I = 0 , \tag{55}$$

where we have expressed all the primed (transformed) fields and charges in terms of the unprimed (original) ones and α is the argument of $c+ie^{-2\phi}$. We have almost recovered the equations of Sec. III. The only evidence of the existence of an axion is the complex function $e^{i\alpha}$.

Equation (53) is solved by

$$\epsilon_I = e^{(1/2)(U+i\alpha)}\hat{\epsilon}_I \ , \tag{56}$$

so the supercovariant spinors are different from those of the original solutions, the difference being the complex (r-dependent) phase $e^{(i/2)\alpha}$. Further comments on the presence of this phase will be made in Sec. VI.

The integrability condition of Eqs. (53)–(55) is again the extremality condition (26). The solution to (53)–(55) is

$$\epsilon_I = e^{(1/2)(U+i\alpha)} e^{(i/2)\gamma^3\gamma^0\theta} e^{(i/2)\gamma^1\gamma^0\varphi} \epsilon_{I_0} \ . \tag{57}$$

Considering now Eqs. (51) and (52), we split them into the 1,2 and 3,4 sectors, etc. (in this case we choose the negative branch of the square root), and we arrive at

$$\begin{aligned} &\partial_r e^{U+\phi}(\epsilon_1 + e^{i\alpha}\xi^*\gamma^0\epsilon^2) = 0 \ , \\ &\partial_r e^{U-\phi}(\epsilon_1 + ie^{i\alpha}\eta^*\gamma^0\epsilon^2) = 0 \ , \end{aligned} \tag{58}$$

for the $I,J=1,2$ sector, and for the $I,J=3,4$ sector we have

$$\begin{aligned} &\partial_r e^{U+\phi}(\epsilon_3 + e^{i\alpha}\xi\gamma^0\epsilon^4) = 0 \ , \\ &\partial_r e^{U-\phi}(\epsilon_3 + ie^{i\alpha}\eta\gamma^0\epsilon^4) = 0 \ . \end{aligned} \tag{59}$$

The discussion of broken and unbroken supersymmetries is the same as we made in Sec. III, including the fact that unbroken supersymmetry implies $\Delta=0$ (not $\Delta'=0$). We will not repeat it here. However, notice that whether a particular supersymmetry is broken or unbroken does not depend on the signs of the *actual* (primed) charged of the solutions but on the *original* (unprimed) charges, which are linear combinations of the actual ones.

VI. CONCLUSIONS

In this paper we have described a new class of solutions to the string effective action with vanishing axion field and found that the extreme ones have unbroken supersymmetries. This provides a more general example in which supersymmetry plays the role of cosmic censor in static asymptotically flat spaces. The related Bogomolnyi-type bound has been displayed, including the axion charge.

Furthermore, we have studied the effect of the SL$(2,R)$ group of electric-magnetic duality rotations that leave the equations of motion invariant on the supersymmetry properties and we have seen that they leave invariant the form of the Bogomolnyi bound (with the axion charge included), giving us a hint that they preserve the unbroken supersymmetries. We have performed explicitly the rotation of this class of solutions under a subset of the whole SL$(2,R)$ generating a nontrivial axion field and learned that the transformed solutions have the same number of unbroken supersymmetries the original solutions had. We conjecture that this will be true for any solution and any SL$(2,R)$ duality rotation.

The presence of the factor $e^{i\alpha}$ in the supercovariantly constant spinors, which is the only trace of the presence of an axion, is a remarkable fact. In Ref. [17], all the backgrounds admitting $N=2$ supercovariantly constant spinors were found. Apart from the plain waves, they were nothing but generalizations of the Israel-Wilson-Perjes [18] class of metrics including charged, rotating dust. They can be described in terms of a single complex function V, from which the (unique) U(1) field is also obtained. V was in the context of [17] the product of the pair of supercovariantly constant SL$(2,C)$ two component spinors, and it would be $e^{U+i\alpha}$ in our case. Some of these metrics admit also $N=4$ supercovariantly constant spinors. Of course, the vector fields are different, and we have to add dilaton and axion. For instance, the metrics of the whole class of extreme electric and magnetic dilaton black holes described in [1] for any value of the parameter a are described in [17]. The vector field has to rescaled by a power of the function V, which is purely real or imaginary in this case, and the dilaton has to be identified with the appropriate function of $V(e^U)$. We have

$$V = e^{\phi/a} \ , \tag{60}$$

$$F^{\text{Tod}} = [\tfrac{1}{2}(1+a^2)]^{-1/2} e^{-a\phi} F^{\text{GHS}} \ , \tag{61}$$

where V has to satisfy the equation

$$\nabla^2 V^{-(1+a^2)} = 0 \ , \tag{62}$$

so $V^{-(1+a^2)}$ is a harmonic function.[5] However, basically only for $a=1$ [3] we can embed these solutions in a theory with local supersymmetry.[6]

In this case V is essentially the dilaton, even though there is no dilaton in $N=2$ supergravity. In the case at hand we have a complex V, the imaginary part reflecting the fact that we have an axion. There is no axion in $N=2$, either, and solutions with complex V in [17] have rotation, in general, and have no diagonal metrics. However, there might be a way of trading rotation by axion, so we can stablish a (formal) connection between two kinds of systems otherwise very different.

We have not explored to its full extent the implications that the preservation of unbroken supersymmetry by SL$(2,R)$ transformations can have if, as conjectured in [4], these are symmetries of the exact string theory and the spectrum of charged black holes is related to the spectrum of excitations of fundamental strings. However, our work seems to support this conjecture by telling us that the supersymmetric structure of the spectrum (supermultiplets) would also be preserved by these transformations. Further work is needed to give a final answer to these problems.

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[5]This permits us to find new multi-black-hole solutions.

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APPENDIX: CONVENTIONS

Here we specify some of the conventions in [3]. Our choice of gamma matrices is

$$\gamma^0=\begin{bmatrix}0 & i\\ i & 0\end{bmatrix}\,,\quad \gamma^i=\begin{bmatrix}0 & -i\sigma^i\\ i\sigma^i & 0\end{bmatrix}\,,$$
$$\gamma_5=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\,,\quad \mathcal{C}=\begin{bmatrix}i\sigma^2 & 0\\ 0 & -i\sigma^2\end{bmatrix}=i\gamma^2\gamma^0\,. \tag{A1}$$

Our choice of vierbiens is

$$e_t^{\,0}=e^{U}\,,\quad e_r^{\,1}=e^{-U}\,,\quad e_\theta^{\,2}=R\,,\quad e_\varphi^{\,3}=R\sin\theta\,, \tag{A2}$$

so the covariant derivative acting on spinors has the components

$$\begin{aligned}
&\nabla_t\epsilon=\tfrac{1}{4}\partial_r(e^{2U})\gamma^1\gamma^0\epsilon\,,\\
&\nabla_r\epsilon=\partial_r\epsilon\,,\\
&\nabla_\theta\epsilon=\partial_\theta\epsilon-\tfrac{1}{2}e^{U}\partial_r R\,\gamma^1\gamma^2\epsilon\,,\\
&\nabla_\varphi\epsilon=\partial_\varphi\epsilon-\tfrac{1}{2}e^{U}\partial_r R\,\sin\theta\gamma^1\gamma^3\epsilon\\
&\qquad-\tfrac{1}{2}e^{U}\cos\theta\gamma^2\gamma^3\epsilon\,.
\end{aligned} \tag{A3}$$

Obviously, t, r, θ, and φ are curved indices, and 0, 1, 2, and 3 are flat.

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